Approximation of Fixed Points of Multivalued Demicontractive and Multivalued Hemicontractive Mappings in Hilbert Spaces

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ABSTRACT: We give control conditions for the approximation of fixed points of multivalued demicontractive and multivalued hemicontractive maps recently introduced in [1]. Many of our conditions are weaker than the conditions used in [1], hence our results improve and complement the convergence results in [1] and the references therein.

Keywords and phrases: Proximinal sets, Hilbert spaces, multivalued k-strictly pseudocontractive-type mappings, multivalued pseudocontractive-type mappings, multivalued demicontractive mappings, multivalued hemicontractive mappings.

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1. INTRODUCTION

Let E be a normed space. A subset K of E is called proximinal if for each $x \in E$ there exists $k \in K$ such that

$$||x-k|| = \inf\{||x-y|| : y \in K\} = d(x, K).$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. In fact if K is a closed and convex subset of a uniformly convex Banach space X, then for any $x \in X$ there exists a unique point $u_x \in K$ such that (see for e.g.[24])

$$||x-u_x|| = \inf\{||x-y|| : y \in K\} = d(x, K)$$

We shall denote the family of all nonempty proximinal subsets of X by $P(X)$, the family of all nonempty closed, convex and bounded subsets of X by $CVB(X)$, the family of all nonempty closed and bounded subsets of X by $CB(X)$ and the family of all nonempty subsets of X by 2^X for a nonempty set X. Let $CB(E)$ be the family of all nonempty closed and bounded subsets of a normed space E. Let H be the Hausdorff metric induced by the metric d on E, that is for every $A, B \in 2^E$,

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

If $A, B \in CB(E)$, then

$$H(A, B) = \inf\{\epsilon > 0 : A \subseteq N(\epsilon, B) \;\; and \;\; B \subseteq N(\epsilon, A)\},$$

where $N(\epsilon, C) = \bigcup_{c\in C} \{x \in E : d(x, c) \leq \epsilon\}$. Let E be a normed space. Let $T : D(T) \subseteq$

$E \to 2^E$ be a multivalued mapping on E. A point $x \in D(T)$ is called a *fixed point* of T if $x \in Tx$. The set $F(T) = \{x \in D(T) : x \in Tx\}$ is called the fixed point set of T. A point $x \in D(T)$ is called a *strict fixed point* of T if $Tx = \{x\}$. The set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of T. A multivalued Mapping $T : D(T) \subseteq E \to 2^E$ is called $L - Lipschitzian$ if there exists $L > 0$ such that for any pair $x, y \in D(T)$

$$H(Tx, Ty) \leq L||x - y||. \tag{1}$$

In (1) if $L \in [0, 1)$ T is said to be a *contraction* while T is *nonexpansive* if $L = 1$. T is called $quasi - nonexpansive$ if $F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for all $p \in F(T)$,

$$H(Tx, Tp) \leq ||x - p||. \tag{2}$$

Clearly every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive. In recent years, much work has been done on the approximation of fixed points of nonlinear multivalued mappings(see e.g. [4], [5], [6], [8], [9], [13], [16] and references therein), using the multivalued equivalence of single valued iterative sequences employed by several authors(see e.g. [10], [11], [12]). Different iterative schemes have been introduced by several authors to approximate the fixed points of multivalued nonexpansive mappings (see for example [4], [5], [6]). Sastry and Babu [3] introduced a Mann-type and an Ishikawa-type iteration schemes as follows:
Let $T : X \to P(X)$ and p be a fixed point of T. The sequence of Mann iterates is generated from an arbitrary $x_0 \in X$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \quad \forall n \geq 0 \tag{3}$$

where $y_n \in Tx_n$ is such that $||y_n - p|| = d(Tx_n, p)$ and α_n is a real sequence in (0,1) $\sum_{n=1}^{\infty} \alpha_n = \infty$.
The sequence of Ishikawa iterates is given by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{4}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such $||z_n - p|| = d(p, Tx_n)$, $||u_n - p|| = d(Ty_n, p)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences satisfying:
(i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n \beta = \infty$.
Using the above iterative schemes, Panyanak [4] generalized the results proved in [3].
Nadler [6] made the following useful Remark, presented as a lemma:

Lemma [6]: let $A, B \in CB(X)$ and $a \in A$. If $\gamma > 0$ then there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \gamma. \tag{5}$$

Using lemma 1, Song and Wang [5] modified the iteration process due to Panyanak [4] and improved the results therein. They made the important observation that generating the Mann and Ishikawa sequences in [3] is in some sense dependent on the knowledge of the fixed point. They gave their iteration scheme as follows:
Let K be a nonempty convex subset of X, Let $\alpha_n, \beta_n \in [0, 1]$ and $\gamma_n \in (0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_1 \in K$, $z_1 \in Tx_1$. Let

$y_1 = (1 - \beta_1)x_1 + \beta_1 z_1$.

Choose $u_1 \in Ty_1$ such that $||z_1 - u_1|| \leq H(Tx_1, Ty_1) + \gamma_1$ and

$x_2 = (1-\alpha_1)x_1 + \alpha_1 u_1.$

Choose $z_2 \in Tx_2$ such that $||z_2 - u_1|| \leq H(Tx_2, Ty_1) + \gamma_2$ and

$y_2 = (1-\beta_2)x_2 + \beta_2 z_2.$

Choose $u_2 \in Ty_2$ such that $||z_2 - u_2|| \leq H(Tx_2, Ty_2) + \gamma_2$ and

$x_3 = (1-\alpha_2)x_2 + \alpha_1 u_2$
Inductively, we have

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{6}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ satisfy $||z_n - u_n|| \leq H(Tx_n, Ty_n) + \gamma_n$, $||z_{n+1} - u_n|| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ and $\{\alpha_n\}$,$\{\beta_n\}$ are real sequences in $[0,1)$ satisfying $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.
Using the above iteration scheme, they proved the following theorem:

Theorem 1(Theorem 1 [5]): Let K be a nonempty compact convex subset of a uniformly convex Banach space X. Suppose that $T : K \to CB(K)$ is a multivalued nonexpansive mapping such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Then the Ishikawa sequence defined as above converges strongly to a fixed point of T.
Shahzad and Zegeye [16] observed that if X is a normed space and $T : D(T) \subseteq E \to P(X)$ is a multivalued mapping then the mapping $P_T : D(T) \to P(X)$ defined for each x by

$$P_T(x) = \{y \in Tx : d(x, Tx) = \|x - y\|\}$$

has the property that $P_T(q) = \{q\}$ for all $q \in F(T)$. Using this idea they removed the strong condition "$T(p) = \{p\}$ for all $p \in F(T)$" introduced by Song and Wang [5].
Khan and Yildirim [17] introduced a new iteration scheme for multivalued nonexpansive mappings using the idea of the iteration scheme for single valued nearly asymptotically nonexpansive mapping introduced by Agarwal et.al [15] as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\lambda)v_n + \lambda u_n \\ y_n = (1-\eta)x_n + \eta v_n, \forall n \in \mathbb{N}. \end{cases} \tag{7}$$

where $v_n \in P_T(x_n)$, $u_n \in P_T(y_n)$ and $\lambda \in [0,1)$. Also, using a lemma in Schu [18], the idea of removal of the condition "$T(p) = \{p\}$ for all $p \in F(T)$" introduced by Shahzad and Zegeye [16], and the method of direct construction of a Cauchy sequence as indicated by Song and Cho [14], they proved the following theorems:

Theorem 2(Theorem 2 [17]: Let X be a uniformly convex Banach space satisfying Opial's condition and K a nonempty closed convex subset of X. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (7). Let $(I - P_T)$ be demiclosed with respect to zero. Then $\{x_n\}$ converges weakly to a point of $F(T)$.

Recently, Isiogugu [2], introduced the new classes of multivalued nonexpansive-type, multivalued k-strictly pseudocontractive-type and multivalued pseudocontractive-type mappings which are more general than the class of multivalued nonexpansive mappings, where:
Definitions 1: Let X be a normed space. A multivalued mapping $T : D(T) \subseteq X \to 2^X$ is said to be *k-strictly pseudocontractive-type* in the sense of Browder and Petryshyn [25]

if there exist $k \in [0,1)$ such that given any $x, y \in D(T)$ and $u \in Tx$, there exits $v \in Ty$ satisfying, $\|u-v\| \leq H(Tx, Ty)$ and

$$H^2(Tx,Ty) \leq ||x-y||^2 + k\|x-u-(y-v)\|^2 \tag{8}$$

If $k = 1$ in (8) T is said to be a *pseudocontractive-type* mapping. T is called multivalued nonexpansive-type if $k = 0$. Clearly, every multivalued nonexpansive mapping is both multivalued k-strictly pseudocontractive-type and multivalued pseudocontractive-type. Using the above definitions, the author proved weak and strong convergence theorems for these classes of multivalued mappings without compactness condition on the domain of the mappings using Mann and Ishikawa iteration schemes.

More recently, Isiogugu and Osilike [1], introduced the new classes of multivalued demicontractive-type and multivalued hemicontractive-type mappings which are more general than the class of multivalued quasi-nonexpansive mappings and which are also related to: the classes of multivalued k-strictly pseudocontractive-type and multivalued pseudocontractive -type mappings introduced in [2], single valued mappings of Browder and Petryshyn [25], Hicks and Kubicek [20]. The authors gave the following definitions:

Definition 2 [1]: Let X be a normed space. A multivalued mapping $T : D(T) \subseteq X \to 2^X$ is said to be demicontractive in the terminology of Hicks and Kubicek [20] if $F(T) \neq \emptyset$ and for all $p \in F(T),\ x \in D(T)$ there exists $k \in [0,1)$ such that

$$H^2(Tx,Ty) \leq ||x-p||^2 + kd^2(x,Tx),$$

where $H^2(Tx,Ty) = [H(Tx,Ty)]^2$ and $d^2(x,p) = [d(x,p)]^2$. If $k = 1$, then T is called hemicontractive.
Using the above definitions, the authors stated and proved the following results:

Theorem 3 [1]: Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is a demicontractive mapping from K into the family of all proximinal subsets of K with $k \in (0,1)$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose $(I-T)$ is weakly demiclosed at zero. Then the Mann type sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n$$

converges weakly to $q \in F(T)$, where $y_n \in Tx_n$ and α_n is a real sequence in $(0,1)$ satisfying: (i) $\alpha_n \to \alpha < 1-k$; (ii) $\alpha > 0$;

Theorem 4 [1]: Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is an L-Lipschitzian hemicontractive mapping from K into the family of all proximinal subsets of K such that $T(p) = \{p\}$ for all $p \in F(T)$. Suppose T satisfies condition (1). Then the Ishikawa sequence defined by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n w_n. \end{cases} \tag{9}$$

converges strongly to $p \in F(T)$, where $u_n \in Tx_n$ and $w_n \in Ty_n$ satisfying the conditions of lemma 3.1 and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying (i) $0 \leq \alpha_n \leq \beta_n < 1$;(ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$; (iii) $\sup_{n\geq 1} \beta_n \leq \beta \leq \frac{1}{\sqrt{1+L^2}+1}$.

Corollary 1 [1]: Let H be a real Hilbert space and K be a nonempty closed and convex subset of H. Suppose that $T : K \to P(K)$ is a k-strictly pseudocontractive-type mapping from K into the family of all proximinal subsets of K with $k \in (0,1)$ such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Suppose $(I - T)$ is weakly demiclosed at zero. Then the Mann sequence $\{x_n\}$ defined in theorem 3.1 converges weakly to a point of $F(T)$.

Corollary 2 [1]: Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $T : K \to P(K)$ be a multivalued mapping from K into the family of all proximinal subsets of K. Suppose P_T is an demicontractive mapping with $k \in (0,1)$ and $(I - P_T)$ is weakly demiclosed at zero. Then the Mann sequence $\{x_n\}$ defined in theorem 3.1 converges weakly to a point of $F(T)$.

Corollary 3 [1]: Let H be a real Hilbert space and K be a nonempty closed and convex subset of H. Suppose that $T : K \to P(K)$ is an L-Lipschitzian pseudocontractive-type mapping from K into the family of all proximinal subsets of K such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Suppose T satisfies condition (1). Then the Ishikawa sequence $\{x_n\}$ defined in (3.10) converges strongly to $p \in F(T)$.

Corollary 4 [1]: Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $T : K \to P(K)$ be a multivalued mapping from K into the family of all proximinal subsets of K such that $F(T) \neq \emptyset$. Suppose P_T is an L-Lipschitzian hemicontractive mapping and $(I - P_T)$. If T satisfies condition (1). Then the Ishikawa sequence $\{x_n\}$ defined in (3.10) converges strongly to $p \in F(T)$.

We observe that many of the conditions on the control sequences for which the above results are proven are strong. For example, the condition $\lim_{n\to\infty} \alpha_n \to \alpha$ imposed on $\{\alpha_n\}$ in theorem 3 is strong. This condition disenfranchises many non-convergent sequences in $(0,1)$, for which the results can be proven. Furthermore, the condition $\sup_{n\geq 1} \beta_n \leq \beta$ in theorem 4 can be dropped and replaced with a milder condition.

It is our purpose in this article to give different sets of control conditions, which are milder than the above condition, under which the results still hold. Furthermore, we drop the condition $\sup_{n\geq 1} \beta_n \leq \beta$ in theorem 4. Some of our computations and analyses pass through the subsequence of a subsequence arguments

Before we proceed to the main results, we state some lemmas and give some definitions which will be useful in the sequel.

Lemma 1 [26]: Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad n \geq n_0,$$

where n_0 is a nonnegative integer. If $\sum b_n < \infty$, $\sum c_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 2 [14]: Let K be a normed space. Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Then the following are equivalent:
(1) $x \in Tx$;
(2) $P_T x = \{x\}$;
(3) $x \in F(P_T)$.
Moreover, $F(T) = F(P_T)$.

Definition 3 [5]: A multivalued mapping $T : K \to P(K)$ is said to satisfy condition

(1) (see for example [4] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$d(x,Tx) \geq f(d(x,F(T)), \ \ \forall x \in K$$

Definition 4(see e.g. [19], [23]: Let E be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multivalued mapping. $I - T$ is said to be *weakly demiclosed* at *zero* if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to p and a sequence $\{y_n\}$ with $y_n \in Tx_n$ for all $n \in \mathbb{N}$ such that $\{x_n - y_n\}$ converges strongly to *zero*. Then $p \in Tp$ (i.e., $0 \in (I-T)p$).

Main Results

Theorem 5: Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is a multivalued demicontractive mapping from K into the family of all proximinal subsets of K with $k \in (0,1)$ such that $T(p) = \{p\}$ for all $p \in F(T)$. Suppose (I-T) is weakly demiclosed at zero. Then the Mann type sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n$$

converges weakly to $q \in F(T)$, where $y_n \in Tx_n$ and α_n is a real sequence in (0,1) satisfying:
(i) $0 < a_1 \leq \alpha_n \leq a_2 < 1-k$, for some $a_1, a_2 \in (0,1)$.

Proof. Using the well known identity

$$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2$$

which holds for all $x, y \in H$ and for all $t \in [0,1]$, $p \in F(T)$ we have

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\alpha_n)x_n + \alpha_n y_n - p||^2 \\
&= ||(1-\alpha_n)(x_n - p) + \alpha_n(y_n - p)||^2 \\
&= (1-\alpha_n)||x_n - p||^2 + \alpha_n||y_n - p||^2 - \alpha_n(1-\alpha_n)||x_n - y_n||^2 \\
&\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n H^2(Tx_n, Tp) - \alpha_n(1-\alpha_n)||x_n - y_n||^2 \\
&\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n\Big[\|x_n - p\|^2 + k||x_n - y_n||^2\Big] \\
&\quad -\alpha_n(1-\alpha_n)||x_n - y_n||^2 \\
&= ||x_n - p||^2 + \alpha_n k||x_n - y_n||^2 - \alpha_n(1-\alpha_n)||x_n - y_n||^2 \\
&= ||x_n - p||^2 - \alpha_n(1-(\alpha_n + k))||x_n - y_n||^2
\end{aligned}$$

It then follows from (i) and lemma 1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence $\{x_n\}$ is bounded. Also,

$$\begin{aligned}
\sum_{n=1}^{\infty} a_1(1-(a_2+k))||x_n - y_n||^2 &\leq \sum_{n=1}^{\infty} \alpha_n(1-(\alpha_n+k))||x_n - y_n||^2 \\
&\leq ||x_0 - p||^2 < \infty.
\end{aligned}$$

This yields $\lim_{n\to\infty} ||x_n - y_n|| = 0$. Also since K is closed and $\{x_n\} \subseteq K$ with $\{x_n\}$ bounded, there exist a subsequence $\{x_{n_t}\}$ of $\{x_n\}$ such that $\{x_{n_t}\}$ converges weakly to some $q \in K$. Also $\lim_{n\to\infty} ||x_n - y_n|| = 0$ implies that $\lim_{t\to\infty} ||x_{n_t} - y_{n_t}|| = 0$. Since $(I-T)$ is weakly demiclosed at *zero* we have that $q \in Tq$. Since H satisfies Opial's condition (see [22]), we have that $\{x_n\}$ converges weakly to $q \in F(T)$. $\square$

Theorem 6: Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is a multivalued demicontractive mapping from K into the family of all proximinal subsets of K with $k \in (0,1)$ such that $T(p) = \{p\}$ for all $p \in F(T)$. Suppose (I-T) is weakly demiclosed at zero. Then the Mann type sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n$$

converges weakly to $q \in F(T)$, where $y_n \in Tx_n$ and α_n is a real sequence in (0,1) satisfying the condition (i) $0 < \alpha_n < 1 - k$, (ii) $0 < \alpha = \liminf \alpha_n$

Proof: Using the well known identity

$$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2$$

which holds for all $x, y \in H$ and for all $t \in [0,1]$, we compute as in theorem 5 to have

$$||x_{n+1} - p||^2 \leq ||x_n - p||^2 - \alpha_n(1 - (\alpha_n + k))||x_n - y_n||^2$$

It then follows from (i) and lemma 1 that $\lim\limits_{n\to\infty} \|x_n - p\|$ exists. Hence $\{x_n\}$ is bounded. Also,

$$\sum_{n=1}^{\infty} \alpha_n(1 - (\alpha_n + k))||x_n - y_n||^2 \leq ||x_0 - p||^2 < \infty.$$

This implies $\lim \alpha_n(1-(\alpha_n+k))||x_n-y_n||^2 = 0$. Using (ii), this yields $\liminf ||x_n-y_n||^2 = 0$. That is $\liminf ||x_n - y_n|| = 0$. This implies there exists a subsequence $\{x_{n_k} - y_{n_k}\}$ of $\{x_n - y_n\}$ such that $\lim ||x_{n_k} - y_{n_k}|| = 0$. Also since K is closed and $\{x_n\} \subseteq K$ with $\{x_n\}$ bounded, then $\{x_{n_k}\}$ is also bounded. Then there exist a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to some $q \in K$. Since $\lim ||x_{n_k} - y_{n_k}|| = 0$, then $\lim\limits_{j\to\infty} ||x_{n_{k_j}} - y_{n_{k_j}}|| = 0$. Since $(I - T)$ is weakly demiclosed at *zero* we have that $q \in Tq$. Since H satisfies Opial's condition (see [22]), we have that $\{x_n\}$ converges weakly to $q \in F(T)$. $\square$

Prototype: A prototype of the sequence $\{\alpha_n\}$ which satisfies the conditions of theorems 5 and 6 is

$$\{\alpha_n\} = \begin{cases} \frac{1}{2}, & if \;\; n \; is \; even \\ \\ \frac{1}{3}, & if \;\; n \; is \; odd \end{cases}$$

Remark: Observe that all the conditions in our theorems do not require that $\{\alpha_n\}$ be a convergent sequence as in [1].

Theorem 7: Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to P(K)$ be an L-Lipschitzian hemicontractive-type mapping from K into the family of all proximinal subsets of K and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose T satisfies condition (1). Then the Ishikawa sequence defined by

$$y_n = (1 - \beta_n)x_n + \beta_n u_n$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n w_n,$$

converges strongly to $p \in F(T)$, where $u_n \in Tx_n$, $w_n \in Ty_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences satisfying (i) $0 \leq \alpha_n \leq \beta_n \leq \frac{1}{\sqrt{1+L^2}+1}$ (ii) $0 < \alpha = \liminf \alpha_n = \liminf \beta_n$

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Proof: Using the well known identity
$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2$
which holds for all $x, y \in H$ and for all $t \in [0,1]$, $p \in F(T)$ we have

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\alpha_n)x_n + \alpha_n w_n - p||^2 \\
&= ||(1-\alpha_n)(x_n - p) + \alpha_n(w_n - p)||^2 \\
&= (1-\alpha_n)||x_n - p||^2 + \alpha_n||w_n - p||^2 \\
&\quad -\alpha_n(1-\alpha_n)||x_n - w_n||^2 \\
&\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n H^2(Ty_n, Tp) \\
&\quad -\alpha_n(1-\alpha_n)||x_n - w_n||^2 \\
&\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n\Big[||y_n - p||^2 \\
&\quad +||y_n - w_n||^2\Big] - \alpha_n(1-\alpha_n)||x_n - w_n||^2 \\
&= (1-\alpha_n)||x_n - p||^2 + \alpha_n||y_n - p||^2 + \alpha_n||y_n - w_n||^2 \\
&\quad -\alpha_n(1-\alpha_n)||x_n - w_n||^2 \qquad (10)
\end{aligned}$$

Also,

$$\begin{aligned}
||y_n - w_n||^2 &= ||(1-\beta_n)x_n + \beta_n u_n - w_n||^2 \\
&= ||(1-\beta_n)(x_n - w_n) + \beta_n(u_n - w_n)||^2 \\
&= (1-\beta_n)||x_n - w_n||^2 + \beta_n||u_n - w_n||^2 \\
&\quad -\beta_n(1-\beta_n)||x_n - u_n||^2 \qquad (11)
\end{aligned}$$

(10) and (11) imply that

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n||y_n - p||^2 \\
&\quad +\alpha_n\Big[(1-\beta_n)||x_n - w_n||^2 + \beta_n||u_n - w_n||^2 \\
&\quad -\beta_n(1-\beta_n)||x_n - u_n||^2\Big] \\
&\quad -\alpha_n(1-\alpha_n)||x_n - w_n||^2 \qquad (12)
\end{aligned}$$

$$\begin{aligned}
||y_n - p||^2 &= ||(1-\beta_n)x_n + \beta_n u_n - p||^2 \\
&= ||(1-\beta_n)(x_n - p) + \beta_n(u_n - p)||^2 \\
&= (1-\beta_n)||x_n - p||^2 + \beta_n||u_n - p||^2 \\
&\quad -\beta_n(1-\beta_n)||x_n - u_n||^2 \\
&\leq (1-\beta_n)||x_n - p||^2 + \beta_n H^2(Tx_n, Tp) \\
&\quad -\beta_n(1-\beta_n)||x_n - u_n||^2 \\
&\leq (1-\beta_n)||x_n - p||^2 + \beta_n\Big[||x_n - p||^2 + ||x_n - u_n||^2\Big] \\
&\quad -\beta_n(1-\beta_n)||x_n - u_n||^2 \\
&= (1-\beta_n)||x_n - p||^2 + \beta_n||x_n - p||^2 + \beta_n||x_n - u_n||^2 \\
&\quad -\beta_n(1-\beta_n)||x_n - u_n||^2 \\
&= ||x_n - p||^2 + \beta_n^2||x_n - u_n||^2. \qquad (13)
\end{aligned}$$

(12) and (13) imply that

$$\begin{aligned}
||x_{n+1}-p||^2 &\le (1-\alpha_n)||x_n-p||^2 \\
&+\alpha_n\Big[||x_n-p||^2+\beta_n^2||x_n-u_n||^2\Big] \\
&+\alpha_n\Big[(1-\beta_n)||x_n-w_n||^2+\beta_n||u_n-w_n||^2 \\
&-\beta_n(1-\beta_n)||x_n-u_n||^2\Big] \\
&-\alpha_n(1-\alpha_n)||x_n-w_n||^2 \\
&= (1-\alpha_n)||x_n-p||^2+\alpha_n||x_n-p||^2+\alpha_n\beta_n^2||x_n-u_n||^2 \\
&+\alpha_n(1-\beta_n)||x_n-w_n||^2+\alpha_n\beta_n||u_n-w_n||^2 \\
&-\alpha_n\beta_n(1-\beta_n)||x_n-u_n||^2-\alpha_n(1-\alpha_n)||x_n-w_n||^2 \\
&\le ||x_n-p||^2+\alpha_n\beta_n^2||x_n-u_n||^2+\alpha_n\beta_nH^2(Tx_n,Ty_n) \\
&-\alpha_n(\beta_n-\alpha_n)||x_n-w_n||^2 \\
&-\alpha_n\beta_n(1-\beta_n)||x_n-u_n||^2 \\
&\le ||x_n-p||^2+\alpha_n\beta_n^2||x_n-u_n||^2+\alpha_n\beta_n^3L^2\|x_n-u_n\|^2 \\
&-\alpha_n\beta_n(1-\beta_n)||x_n-u_n||^2 \\
&-\alpha_n(\beta_n-\alpha_n)||x_n-w_n||^2 \\
&= ||x_n-p||^2-\alpha_n\beta_n[1-2\beta_n-L^2\beta_n^2]||x_n-u_n||^2 \\
&-\alpha_n(\beta_n-\alpha_n)||x_n-w_n||^2 \\
&= ||x_n-p||^2-\alpha_n\beta_n[1-2\beta_n-L^2\beta_n^2]||x_n-u_n||^2 \qquad (14)
\end{aligned}$$

It then follows from condition (i) and Lemma 1 that $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence $\{x_n\}$ is bounded, so also are $\{u_n\}$ and $\{w_n\}$. We have from (14) that

$$\begin{aligned}
\sum_{n=0}^{\infty}\alpha_n\beta_n[1-2\beta_n-L^2\beta_n^2]||x_n-u_n||^2 &\le \sum_{n=0}^{\infty}[||x_n-p||^2-||x_{n+1}-p||^2] \\
&\le ||x_0-p||^2+D<\infty
\end{aligned}$$

This implies that $\lim\alpha_n\beta_n[1-2\beta_n-L^2\beta_n^2]||x_n-u_n||^2=0$. It then follows from (ii) that $\liminf||x_n-u_n||=0$. Then there exists a subsequence $\{x_{n_j}-u_{n_j}\}$ of $\{x_n-u_n\}$ such that $\lim||x_{n_j}-u_{n_j}||=0$. Since $u_{n_j}\in Tx_{n_j}$ we have that $d(x_{n_j},Tx_{n_j})\le||x_{n_j}-u_{n_j}||\to 0$ as $j\to\infty$. Since T satisfies condition (1) then $\lim_{j\to\infty}d(x_{n_j},F(T))=0$. Thus there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ such that $\|x_{n_{j_k}}-p_k\|\le\frac{1}{2^k}$ for some $\{p_k\}\subseteq F(T)$. From (14)

$$\|x_{n_{j_{k+1}}}-p_k\|\le\|x_{n_{j_k}}-p_k\|.$$

We now show that $\{p_k\}$ is a Cauchy sequence in $F(T)$.

$$\begin{aligned}
\|p_{k+1}-p_k\| &\le \|p_{k+1}-x_{n_{j_{k+1}}}\|+\|x_{n_{j_{k+1}}}-p_k\| \\
&\le \frac{1}{2^{k+1}}+\frac{1}{2^k} \\
&\le \frac{1}{2^{k-1}}
\end{aligned}$$

Therefore $\{p_k\}$ is a Cauchy sequence and converges to some $q \in K$ since K is closed. Now,

$$\|x_{n_{j_k}} - q\| \leq \|x_{n_{j_k}} - p_k\| + \|p_k - q\|$$

Hence $x_{n_{j_k}} \to q$ as $k \to \infty$.

$$\begin{aligned} d(q, Tq) &\leq \|q - p_k\| + \|p_k - x_{n_{j_k}}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \\ &\leq \|q - p_k\| + \|p_k - x_{n_{j_k}}\| + d(x_{n_{j_k}}, Tx_{n_{j_k}}) + L\|x_{n_{j_k}} - q\| \end{aligned}$$

Hence $q \in Tq$. Since $\lim \|x_n - q\|$ exists and $\{x_{n_{j_k}}\}$ converges strongly to q, we have that $\{x_n\}$ converges strongly to $q \in F(T)$.

Corollary 5: Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $T : K \to P(K)$ be a multivalued mapping from K into the family of all proximinal subsets of K such that $F(T) \neq \emptyset$. Suppose P_T is an L-Lipschitzian hemicontractive mapping. If T satisfies condition (1). Then the Ishikawa sequence $\{x_n\}$ defined in theorem 7 converges strongly to $p \in F(T)$.

Proof: The proof follows easily from Lemma 2 and Theorem 7. □

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